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Diffusion of thermal disturbances in two-dimensional Cartesian transient heat conduction

Filippo de Monte [a,*], James V. Beck [b], Donald E. Amos [c]

[a] Dipartimento di Ingegneria Meccanica, Energetica e Gestionale, University of L'Aquila, Località Monteluco, 67040 Roio Poggio, L'Aquila, Italy
[b] Department of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA
[c] Sandia National Laboratories, Albuquerque, NM 87110, USA

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ABSTRACT

This paper analyzes the diffusion of thermal disturbances in heat-conducting two-dimensional rectangular bodies through characteristic times, such as penetration and deviation times, denoting their effects within a certain order of magnitude. A single basic criterion governing the above diffusion is derived thanks to the similarity of the findings. It allows very accurate solutions to be obtained considering in advance only the physical region of interest in place of considering the complete body. Therefore, it is efficient in terms of modeling and computational effort in numerically based methods as well as analytical techniques. In the former case, the grid domain can considerably be reduced. In the latter case, the number of terms needed to obtain long-time solutions when time-partitioning is applied can significantly be limited. Also, complex 1D and 2D semi-infinite problems are solved explicitly in the paper and evaluated numerically as part of the analysis.

1. Introduction

1.1. Problem description

In unsteady heat conduction, early thermal disturbances inside a solid regular in shape can essentially be due to: (1) a heating process (through a boundary surface or a heat source) and (2) a homogeneous boundary condition (where the boundary can be parallel or perpendicular to the heated surface). The former may be indicated as 'active' disturbance and it is responsible of thermal penetration effects. The latter is, however, an 'inactive' disturbance (because it is strictly related to the 'active' one) and it can cause thermal deviation effects.

The analysis of the diffusion of the above thermal disturbances inside a solid is an important step because it can in advance give insight into how long their effects can be considered negligible (or within a certain order of magnitude) in a region of the same solid. This would indicate that the complete domain need not be considered to obtain accurate solutions in a sub-domain of interest, depending upon the times of interest. Considerable advantage can be derived from this to saving computations in finite difference, finite control volume, finite element, etc., methods when a semi-infinite or "large" body is considered. However, there can be advantages for analytical methods too. For instance, the number of terms in 2D summations of long-time solutions may substantially be reduced for large aspect ratios when only part of the problem need be considered.

1.2. Literature review

In fluid mechanics and convection, thermal penetration effects in a fluid at a uniform temperature flowing along a body maintained at a different constant temperature are well-known through the concept of thermal boundary layer thickness [1]. It is defined as the distance from the body surface where the fluid temperature deviates by 1% from its free-stream temperature. Also, it may be defined as the distance where the fluid temperature is affected at a level of 0.01 by a heating at its boundary surface.

It is relevant to note that the above definition of thermal boundary layer thickness deals with only the differential formulation of the heat convection problem, by which exact analytical solutions can be obtained. However, if our primary interest lies rather in the integral formulation of the problem (first used by von Karman and Pohlhausen in 1921 [1]), by which approximate analytical solutions can be derived, this requires that the thermal boundary layer thickness be defined in a different way, that is, as the distance beyond which the fluid temperature is not affected at all by the heating at its boundary surface.

* Corresponding author. Tel.: +39 0862 434326; fax: +39 0862 434303.
E-mail addresses: demonte@ing.univaq.it (F.de Monte), beck@egr.msu.edu (J.V. Beck), deamos@swcp.com (D.E. Amos).

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Nomenclature

B_y	Biot number for 1D case, hy/k
B_{xy}	Biot number for 2D case, $h\sqrt{x^2+y^2}/k$
d	distance
g	volume energy generation
G	Green's function (subscript designates the boundary conditions)
h	heat transfer coefficient at the boundary $y = 0$ of Fig. 1 ($h = h_{y=0}$)
k	thermal conductivity
P	space coordinate ratio, x/y ($P = \cot\theta$)
q	heat flux
t	time coordinate
t^+_{xy}	dimensionless time for 2D case, $\alpha t/(x^2 + y^2)$
t^+_x	dimensionless time for 2D case when $y = 0$ (or $P \to \infty$), $\alpha t/x^2$
t^+_y	dimensionless time for 1D case; and 2D case when $x = 0$ (or $P = 0$), $\alpha t/y^2$
T	temperature
u	cotime
x, y	space coordinates
α	thermal diffusivity
ε	dimensionless group estimating the thermal deviation effects
θ	polar coordinate ($\theta = \cot^{-1}P$)
σ	dimensionless group estimating the thermal penetration effects

Superscripts

q	heat flux
T	temperature
x, y	in the x- and y-directions

The concept of thermal boundary layer thickness is equally attractive for solving transient problems governed by a diffusion type equation. However, it was defined only in the integral formulation of the problem (first used by Biot in 1957 [2]), by which approximate analytical solutions can be derived, and it was called penetration depth. Hence, when the primary interest lies in the differential formulation, by which exact analytical solutions can be obtained, its definition is somewhat neglected in heat conduction [3–20]. For example, Myers [7, p. 222] as well as Taler and Duda [19, p. 375] just give a relatively crude definition of thermal penetration depth and they do not seem to mention any application of this concept.

Recently, however, Tarn and Wang [21] have studied the steady state diffusion of prescribed arbitrary end conditions in circular cylinders through a characteristic decay length. It is defined as the distance measured from the end beyond which the temperature and heat flux reduce to 1% of their values on the end. Using the above time-independent length, the end effects may be confined to a local region near the ends provided boundary conditions of the first kind are applied to the lateral surface. Thus, the thermal field may be evaluated in a sub-region of the entire solid where 2D solutions in place of 3D ones can be used with the accuracy stated above.

1.3. Outline

The diffusion of thermal disturbances has been analyzed in the present work for a two-dimensional Cartesian solid by defining some characteristic times, such as penetration and deviation times. They account for the penetration and deviation effects caused by the above disturbances within a certain order of magnitude. In transient heat conduction with heating at a boundary, for instance, the temperature rise diffuses slowly into the solid with the surface changing quickly and interior points experiencing what seems to be a slow penetration. Also, there seems to be a slow deviation for the 2D case to change from the 1D one.

In particular, the penetration time comes from a 1D concept and it is defined as the time that it takes for the temperature (or heat flux) at an interior point to be just affected by either a heating at the boundary surface (Section 2) or a heat source (Section 3). The deviation time due to a homogeneous boundary parallel and opposite to the heating one derives in a straightforward manner from the previous concept of 1D-penetration time. It may be defined as the time that it takes for the temperature (or heat flux) at a point in a 1D finite solid heated at a boundary to be just affected by the presence of the homogeneous boundary condition (Section 4). The deviation time due to a homogeneous boundary perpendicular to the heating one, however, comes from a 2D concept. It is defined as the time that it takes for the thermal deviation effects due to the homogeneous boundary to just begin to be significant for a point inside the body (Sections 5–8). Similarly, the 2D deviation time due to a homogeneous boundary 'parallel' and 'coplanar' to the heating one is defined (Section 9).

For the above purposes, complex 1D and 2D semi-infinite problems have been solved explicitly using Green's functions and routinely evaluated numerically as integrating part of the analysis (Sections 3, 6 and 7). By "just affected" or "just begin to be significant" we mean to some sufficiently small numerical value such as 10^{-2} (typical of thermal boundary layer thickness) or even the much smaller value of 10^{-10}. The purpose of 10^{-2} is for engineering insight and visual comparisons. The 10^{-10} is for verification purposes of large multi-dimensional numerical codes [22–25] and related intrinsic verification methods [26].

The above characteristic times have been derived in the paper with reference to 1D and 2D Cartesian solids heated either through the boundary at constant temperature, heat flux or ambient temperature or through local heat sources. Heat pulses at time zero have been considered too. Using these results as a model, a single basic criterion able to model the diffusion of thermal disturbances inside a 2D heat-conducting body has been defined in virtue of the amazing similarity of the results (Section 10). It gives the dimensionless time $(\alpha t_{\text{dist}}/d^2) = 0.1/n$ that it takes for a generic thermal disturbance at a point of a solid to reach another point of the same solid at the level of one part in 10^n. The d denotes the distance between the two points. In detail, the location of the thermal disturbance is the actual location for an 'active' disturbance but it is the 'virtual' location for an 'inactive' one (Sections 4, 8 and 9). Not only is the criterion insensitive to various boundary conditions and 2D conditions, but it is also relatively insensitive to the level desired for the thermal disturbance effects. Decreasing, in fact, deviation from 10^{-2} to 10^{-10}, a factor of 10^8 results in a decrease in the criterion only be a factor of 5.

2. Penetration time due to a heating process at a boundary

To obtain the penetration time, consider a homogeneous 1D semi-infinite body $y \geqslant 0$, initially at zero temperature and with temperature-independent properties, subject to a step change in

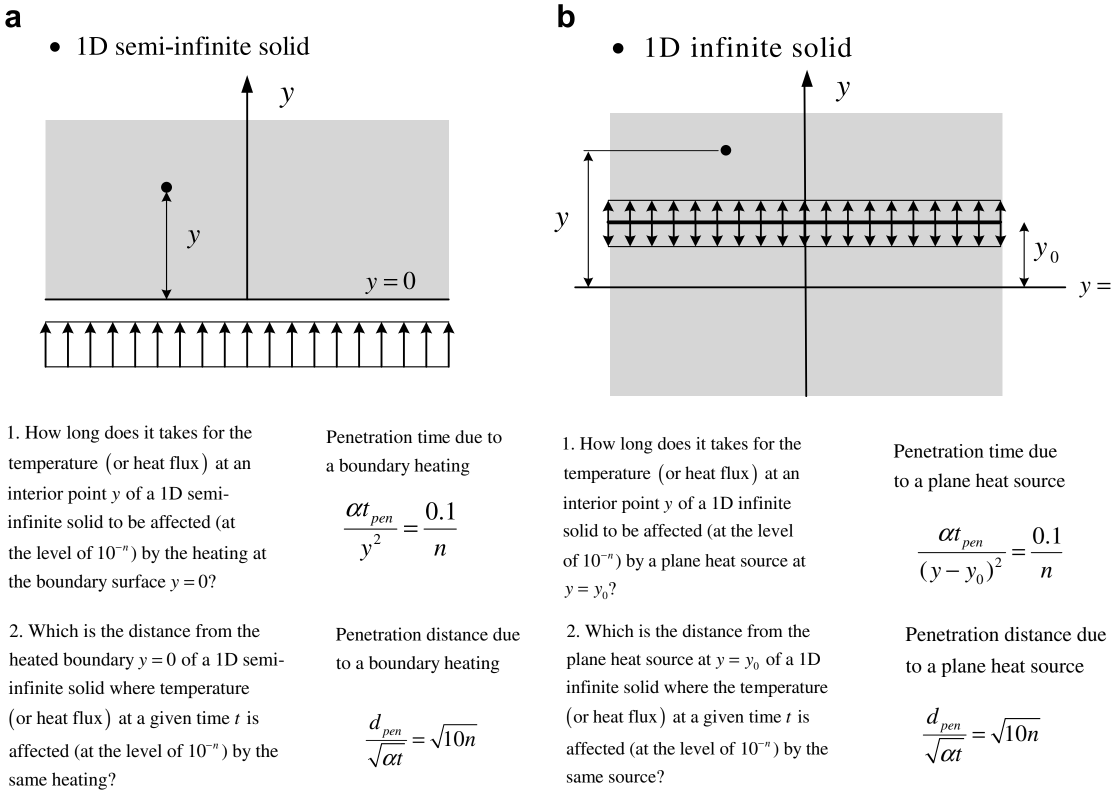

Fig. 1. Penetration times and distances due to a heating process. a) heating at the boundary $y = 0$; b) plane heat source at $y = y_0$.

either temperature, or heat flux or ambient temperature at $y = 0$, as shown in Fig. 1a. This transient heat conduction problem may concisely be denoted by *YI0B1T0* ($I = 1$, 2 or 3), where *Y* denotes the *y*-direction; *I* denotes the kind of boundary condition at $y = 0$; and *T0* would indicate a zero initial temperature (see more detail in Ref. [10, chap. 2] for the numbering system devised by Beck, et al., where for 1D problems, however, the *x*-coordinate was considered leading to the denotation *X*). The thermal penetration effects due to the heated (or cooled) boundary surface $Y = 0$ may be estimated through the ratios of the temperature, *t*, and heat flux, *q*, by

$$\sigma_T(y,t) = \frac{T_{YI0B1T0}(y,t)}{T_{YI0B1T0}(0,t)} \quad \sigma_q(y,t) = \frac{q_{YI0B1T0}(y,t)}{q_{YI0B1T0}(0,t)} \tag{1}$$

where σ_T and σ_q are less than 1 and the 1D semi-infinite solutions are given in Ref. [10, chap. 6]. The above effects as well as the penetration times deriving from them are practically insensitive to the boundary condition types ($I = 1$, 2 or 3), as shown in the following.

Boundary condition of the 3rd kind. In this case, the non-zero boundary condition is

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} + hT(y=0,t) = f(y=0) \tag{2}$$

where h is the heat transfer coefficient and $f(y = 0)$ is usually equal to hT_∞ with T_∞ being the ambient temperature. However, $f(y = 0)$ can also include a prescribed heat flux. The well-known solution is

$$\begin{aligned} T(y,t) &= \frac{f(y=0)}{h}\left[\operatorname{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right) - U\left(\frac{h\sqrt{\alpha}}{k}, \frac{y}{\sqrt{4\alpha}}, t\right)\right] \\ q(y,t) &= f(y=0)U\left(\frac{h\sqrt{\alpha}}{k}, \frac{y}{\sqrt{4\alpha}}, \right) \end{aligned} \tag{3}$$

where the *U*-function may be taken as

$$U\left(\frac{h\sqrt{\alpha}}{k}, \frac{y}{\sqrt{4\alpha}}, t\right) = e^{\frac{hy}{k} + \left(\frac{h\sqrt{\alpha t}}{k}\right)^2} \operatorname{erfc}\left(\frac{y}{\sqrt{4\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right) \tag{4}$$

Substituting Eq. (3) in Eq. (1) and using the two dimensionless variables $t_y^+ = \alpha t/y^2$ and $B_y = hy/k$, we obtain

$$\begin{aligned} \sigma_T(t_y^+, B_y) &= \frac{\operatorname{erfc}\left(\frac{1}{\sqrt{4t_y^+}}\right) - U\left(\frac{B_y\sqrt{t_y^+}}{\sqrt{t}}, \frac{\sqrt{t}}{\sqrt{4t_y^+}}, t\right)}{1 - U\left(\frac{B_y\sqrt{t_y^+}}{\sqrt{t}}, 0, t\right)} \\ \sigma_q(t_y^+, B_y) &= U\left(\frac{B_y\sqrt{t_y^+}}{\sqrt{t}}, \frac{\sqrt{t}}{\sqrt{4t_y^+}}, t\right) \Big/ U\left(\frac{B_y\sqrt{t_y^+}}{\sqrt{t}}, 0, t\right) \end{aligned} \tag{5}$$

The thermal penetration effects σ_T and σ_q depend on only two variables, i.e. $t_y^+ = \alpha t/y^2$ and $B_y = hy/k$. Results in terms of $t_{y,\text{pen}}^+$ from the numerical solution of Eq. (5) for different values of B_y and σ are given in Table 1.

Boundary condition of the 1st kind. When $f(y = 0) = hT_\infty$ and $h \to \infty$, Eq. (2) reduces to a boundary condition of the first kind, that is, $T(y = 0,t) = T_\infty$. In such a case, Eq. (5) reduce to

Table 1
Dimensionless penetration times $t_{y,\text{pen}}^+$ for the thermal penetration effects σ_T and σ_q to reach different small numbers 10^{-n} for different values of B_y (*Y30B1T0* case)

	Thermal penetration effects on temperature (σ_T)			Thermal penetration effects on heat flux (σ_q)		
B_y	10^{-2}	10^{-4}	10^{-10}	10^{-2}	10^{-4}	10^{-10}
0	0.0970	0.0387	0.0130	0.0754	0.0330	0.0120
10^{-2}	0.0970	0.0387	0.0130	0.0754	0.0330	0.0120
1	0.0936	0.0382	0.0129	0.0720	0.0325	0.0119
10^2	0.0764	0.0336	0.0121	0.0554	0.0276	0.0110
∞	0.0754	0.0330	0.0120	0.0543	0.0271	0.0109

$$\sigma_T(t_y^+) = \mathrm{erfc}\left(\frac{1}{\sqrt{4t_y^+}}\right) \quad \sigma_q(t_y^+) = \mathrm{e}^{-\frac{1}{4t_y^+}} \tag{6}$$

Results from the numerical solution of Eq. (6) are given in Table 1 for $B_y \to \infty$. As a matter of fact, the results for $\sigma_q = 10^{-n}$ can be obtained by analytically solving the second of the two Eq. (6) for the dimensionless time $t^+_{y,\mathrm{pen}}$. In fact, it results in: $t^+_{y,\mathrm{pen}} \cong 0.1/n$, where 0.1 is a conservative value for $[4\ln(10)]^{-1} \cong 0.109$.

Boundary condition of the 2nd kind. When $h \to 0$, Eq. (2) reduces to a boundary condition of the second kind (prescribed heat flux $f(y = 0)$ at $y = 0$). In such a case, Eq. (5) become

$$\sigma_T(t_y^+) = \sqrt{\pi}\,\mathrm{ierfc}\left(\frac{1}{\sqrt{4t_y^+}}\right) \quad \sigma_q(t_y^+) = \mathrm{erfc}\left(\frac{1}{\sqrt{4t_y^+}}\right) \tag{7}$$

Results in terms of $t^+_{y,\mathrm{pen}}$ for different values of $\sigma = 10^{-n}$ may be obtained numerically solving Eq. (7). They are given in Table 1 for $B_y = 0$.

Summary of the YI0B1T0 problems. The important and amazing point is that there is a relatively little difference for the penetration times when B_y varies from zero to infinity (Table 1). For $\sigma = 10^{-10}$, in fact, the dimensionless penetration times varies only from 0.0130 down to 0.0120 for the temperature, a reduction of only 7.7 %, and from 0.0120 down to 0.0109 for the heat flux, a reduction of only 9.2%. Also, decreasing the penetration effects on temperature from 10^{-2} to 10^{-10}, a factor of 10^{-8}, for $B_y = 1$ results in a decrease in the penetration time only from 0.0936 to 0.0129, that is only a factor of 7.

Therefore, for the different cases denoted *YI0B1T0*, *I* = 1, 2 or 3, and for the temperature and heat flux at a point *y* to rise about one part in 10^n with respect to those at $y = 0$, a estimate for the penetration time is about

$$\frac{\alpha t_{\mathrm{pen}}}{y^2} \cong \frac{0.1}{n} \quad (y \geqslant 0, t > 0) \tag{8}$$

where y^2 is the square of the distance between the 'active' thermal disturbance located at $y = 0$ and the point of interest *y*, as shown in Fig. 1a. When $n = 2$, we have the 'visual' penetration time. Now, as the most rapid variation would be for an uniform sudden change in temperature at the heated surface $y = 0$ (most conservative case), Eq. (8) gives the smallest penetration time. Therefore, it covers any time and space variation of temperature, heat flux or ambient temperature at the boundary $y = 0$ (for a given maximum surface temperature, heat flux or ambient temperature), including the heat pulse at time zero, as shown in Subsection 2.1.

If we take the reciprocal of the square root of the penetration time (8) we get a conservative estimate for the penetration distance d_{pen}, that is, the distance from the heated boundary at which temperature and heat flux are just affected at a given time *t* by this heating (Fig. 1a). Thus, we have

$$\frac{d_{\mathrm{pen}}}{\sqrt{\alpha t}} \cong \sqrt{10n} \quad (y \geqslant 0, t > 0) \tag{9}$$

Decreasing the penetration effects from 10^{-2} to 10^{-10} results in an increase in the penetration distance from about 4.5 to 10 (only a factor of 2), where the constant 4.5 is not far from the well-known constant 5 occurring in the laminar boundary layer thickness [1].

2.1. Y20Bt7T0 problem

In the case of a heat pulse at time zero and $y = 0$, the applied surface heat flux follows the law $Q_0\delta(t)$ (where Q_0 has units of J/m^2). This transient problem may be denoted as *Y20Bt7T0*, where *t*7 indicates the Dirac delta function $\delta(t)$ which has the units of s^{-1}. In such a case, the temperature solution may be evaluated using Green's functions approach

$$T(y,t) = Q_0\frac{\alpha}{k}\int_{u=0}^{t}\delta(t-u)G_{Y20}(y,0,u)\mathrm{d}u \tag{10}$$

where the $G_{YI0}(y,y',u)$ function (I = 1 or 2) is given by [10, p. 80]

$$G_{YI0}(y,y',u) = K(y-y',u) + (-1)^I K(y+y',u)$$
$$K(y \pm y',u) = \frac{1}{\sqrt{4\pi\alpha u}}\mathrm{e}^{-\frac{(y\pm y')^2}{4\alpha u}} \tag{11}$$

In Eqs. (10) and (11) we have the variable $u \equiv t - \tau$, which we shall call the "cotime." This variable might be also called the convolution or reverse time but, for simplicity, we will call it the cotime. Substituting Eqs. (11) for $I = 2$ and $y' = 0$ in Eq. (10), we simply have

$$T(y,t) = \frac{Q_0\alpha}{k}\frac{1}{\sqrt{\pi\alpha t}}\mathrm{e}^{-\frac{y^2}{4\alpha t}} \tag{12}$$

The ratio of the temperature to the temperature at $y = 0$ gives the thermal penetration effects σ_T due to the heat pulse at time zero and $y = 0$. We obtain the same result as the second expression in Eq. (6) which estimates the thermal penetration effects σ_q for the *Y10B1T0* case (Table 1 for $B_y \to \infty$).

3. Penetration time due to a heating process through a local heat source

To obtain the penetration time, consider an infinite body $-\infty < y < \infty$ with properties independent of temperature and position. As the initial condition is considered homogeneous, we assume that the transient variation in the temperature is caused by a plane heat source (or heat sink) located at $y = y_0$ (Fig. 1b), which is the most conservative heat source between local (point, line and plane) and volume ones. In such a case, the volume heat generation is

$$g(y,t) = q_{y_0}\delta(y-y_0)g_t(t) \tag{13}$$

where q_{y_0} has the units of W/m^2; $\delta(y - y_0)$ is the Dirac delta function (m^{-1}); and $g_t(t)$ is a dimensionless arbitrary function in time. As q_{y0} is independent of the coordinates *x* and *z*, we have a 1D problem.

This transient heat conduction problem may concisely be denoted by *Y00T0Gy7t-*, where *Gy*7 denotes the plane source and *t*− would indicate an arbitrary function in time [10, chap. 2]. Its temperature solution can be obtained using the GFSE (Green's function solution equation), that is,

$$T(y,t) = q_{y0}\frac{\alpha}{k}\int_{u=0}^{t}g_t(t-u)\int_{y'=-\infty}^{\infty}G_{Y00}(y,y',u)\delta(y'-y_0)\mathrm{d}u \tag{14}$$

where the GF on the RHS is given by only $K(y - y',u)$ of Eq. (11). In view of Table 5.1 of Ref. [10], the integration with respect to the dummy variable y' in Eq. (14) is a simple matter. Thus, we have

$$T(y,t) = q_{y0}\frac{\alpha}{k}\int_{u=0}^{t}g_t(t-u)K(y-y_0,u)\mathrm{d}u \tag{15}$$

The thermal penetration effects due to the plane heat source (13) may be estimated through the ratios of the temperature, *T*, and heat flux, *q*, as done in Section 2 by means of Eq. (1) provided the denominators are computed at $y = y_0$. These effects as well as the penetration times deriving from them are practically insensitive to the form of the $g_t(t)$ function:

- For a step change at $t = 0$ (i.e. $g_t(t) = 1$ – Heaviside unit step function), the integral appearing in Eq. (15) is given in Table 5.4 of Ref. [10] as integral 1. It will result in

$$T(y,t) = \frac{q_{y0}}{k}\sqrt{\alpha t}\,\mathrm{ierfc}\left(\frac{|y-y_0|}{\sqrt{4\alpha t}}\right) \tag{16}$$

Notice that Eq. (16) is symmetric about $y = y_0$ and the maximum temperature is finite and occurs at $y = y_0$. Also, Eq. (16) states that the $Y00T0Gy7t1$ case with $y_0 = 0$ is the same as the case denoted by $Y20B1T0$ (Section 2) apart from a factor of 2 accounting for the half-space vs. the whole space (i.e. $T_{Y20} = 2T_{Y00}$). Similar considerations are hence valid for the 1D-penetration times (Table 1 for $B_y \to 0$):

- For $f(t) = \sqrt{t_0/t}$, where t_0 is some reference time, the integral appearing in Eq. (15) is given in Table 5.5 of Ref. [10] as integral 1. In such a way, we obtain the temperature solution $(T_0/2)\mathrm{erfc}\left(|y-y_0|/\sqrt{4\alpha t}\right)$ where $T_0 = (q_0/k)\sqrt{\alpha t_0/\pi}$. For $y_0 = 0$, we have the same result of the $Y10B1T0$ problem treated in Section 2 apart from a factor of 2. Therefore, similar conclusions may be drawn for the thermal penetration effects (Table 1 for $B_y \to \infty$).
- For a heat pulse at $t = 0$ (i.e. $q_{y0}g_t(t) = Q_{y0}\delta(t)$, where Q_{y0} has the units of J/m^2), the solution (15) provides exactly the temperature (12) divided by 2 when $y_0 = 0$.

From what has been said, it follows that for the cases denoted $Y00T0Gy7t-$ and for the temperature at an arbitrary plane y to rise about one part in 10^{-n} with respect to that at the plane $y = y_0$ where the heat source is located, the smallest 1D-penetration time is about $0.1/n$. Similarly, for the heat flux. A estimate for the penetration time is in general

$$\frac{\alpha t_{\mathrm{pen}}}{(y-y_0)^2} \cong \frac{0.1}{n} \quad (y \geqslant 0, t > 0) \tag{17}$$

where $(y-y_0)^2$ is the square of the distance from the 'active' thermal disturbance located at $y = y_0$ and the point of interest y, as shown in Fig. 1b. Also, a estimate for the penetration distance may be taken as $d_{\mathrm{pen}}/\sqrt{\alpha t} \cong \sqrt{10n}$. As the most rapid variation in temperature at $y = y_0$ for the same maximum temperature rise would be for a uniform source term proportional to $1/\sqrt{t}$ at the same plane (most conservative case), Eq. (17) gives the smallest penetration time. Therefore, it covers any time and space variation of the heat source at this plane including the case of a heat pulse at $t = 0$.

4. Deviation time due to a homogeneous boundary parallel and opposite to the heating surface

To obtain the deviation time, consider a 1D finite Cartesian body $0 \leqslant y \leqslant W$ with temperature-independent properties and initially at a uniform temperature (which without loss of generality, we can assume to be zero). At time $t = 0$, heating (or cooling) begins at $y = 0$ and it can be of any kind (1st, 2nd or 3rd), as depicted in Fig. 2a.

Three locations are here of interest. The first is the location of the heating which is $y = 0$. The second location is where the temperature just begins to have an effect due to the heating at $y = 0$; let us call this location W. The third location is the point y of interest. At this point we wish to know the thermal deviation effects due to the boundary condition at $y = W$. This boundary has to be "homogeneous" (with conditions of the 1st, 2nd or 3rd kinds), that is, not introducing heating or cooling at that surface until the effect of the $y = 0$ reaches there. These effects may be estimated through the ratios of the temperature, T, and heat flux, q, by

$$\varepsilon_T(y,t) = \left|\frac{T_{YIJB10T0}(y,t) - T_{YI0B1T0}(y,t)}{T_{YI0B1T0}(0,t)}\right|$$
$$\varepsilon_{q_y}(y,t) = \left|\frac{q_{y,YIJB10T0}(y,t) - q_{y,YI0B1T0}(y,t)}{q_{y,YI0B1T0}(0,t)}\right| \tag{18}$$

where the 1D finite solutions of the $YIJB10T0$ problems (I, J = 1, 2 or 3) are given in Ref. [10, chap. 6]. The above effects as well as the 1D-deviation times deriving from them are practically insensitive to the boundary condition types (I, J = 1, 2 or 3).

Then, the deviation time, that is, the time that it takes for the temperature and heat flux to be just disturbed by the homogeneous boundary condition at $y = W$ and parallel to the heating surface, may be found using the concept of 1D-penetration time defined previously (Section 2). For that purpose, in fact, we can think of the $YIJB10T0$ conductive problem now being composed of a plate $2W$ thick and boundary conditions similar (or opposite) to those at $y = 0$ imposed at $y = 2W$. In detail, the homogeneous boundary condition of the first kind at $y = W$ can be simulated by having a non-zero boundary term at $y = 2W$ that is the negative of that (1st, 2nd or 3rd kind) at $y = 0$. The first kind of homogeneous boundary condition is the same as an infinite heat transfer coefficient ($h \to \infty$) in Eq. (2) rewritten for $f(y = 0) = 0$. If we have a second kind of homogeneous boundary condition at $y = W$, it can be simulated by having a non-zero boundary term at $y = 2W$ that is equal to that (1st, 2nd or 3rd kind) at $y = 0$. This condition is the same as a zero heat transfer coefficient ($h \to 0$) in Eq. (2) rewritten for $f(y = 0) = 0$. Hence, the first and second homogeneous boundary conditions include the extremes of the homogeneous form of the convective boundary condition (2), i.e. the third kind. Now, if the point of interest is at $y = 0$, then y in Eq. (8) would be replaced by $2W$ to obtain the penetration time desired, that is, $\alpha t_{\mathrm{pen}}/(2W)^2 = 0.1/n$.

Then, using the concept of plate of thickness $2W$ and symmetric boundary conditions, a conservative estimate for the deviation time is in general

$$\frac{\alpha t_{\mathrm{dev}}}{(2W-y)^2} \cong \frac{0.1}{n} \tag{19}$$

It is relevant to note that, contrary to what happens in the penetration times (8) and (17), the length $(2W - y)$ appearing in the deviation time (19) is greater than the distance between the 'inactive' thermal disturbance (homogeneous boundary condition) at $y = W$ and the point of interest y, namely $(W - y)$, as shown in Fig. 2a. Therefore, this length may be considered as the distance between the 'virtual' location $y = 2W$ of the thermal 'inactive' disturbance and the actual location y of the point of interest.

As the most rapid variation would be for an uniform sudden change in temperature at the heated surface, the deviation time (19) covers any time and space variation of the temperature, heat flux or ambient temperature at this boundary (for a given maximum surface temperature, heat flux or ambient temperature), including the heat pulse at time zero.

Now, the deviation time (19) may be split into three components, 1) the penetration time calculated at $y = W$ (where W is the thickness of the slab), 2) the co-penetration time related the homogeneous boundary condition at $y = W$ and parallel to the heating surface and 3) another time which we shall call the "finite-space time." We believe that it accounts for the finite dimension of the solid. Thus, we have

$$t_{dev} \cong \underbrace{\frac{0.1}{n\alpha}W^2}_{t_{\mathrm{pen}}(W)} + \underbrace{\frac{0.1}{n\alpha}(W-y)^2}_{t_{\mathrm{co\text{-}pen}}(y,W)} + \underbrace{\frac{0.1}{n\alpha}[2W(W-y)]}_{t_{f-sp}(y,W)} \tag{20}$$

Therefore, when the point of interest is located at the heating boundary surface ($y = 0$), the lengths appearing in $t_{co-\mathrm{pen}}$ and t_{f-sp} are W and $W\sqrt{2}$, respectively, where W is the distance between the homogeneous boundary and the point of interest. However, when $y = W$, the co-penetration and 'finite-space' times vanish as well as the relative thermal co-penetration and 'finite-space' effects.

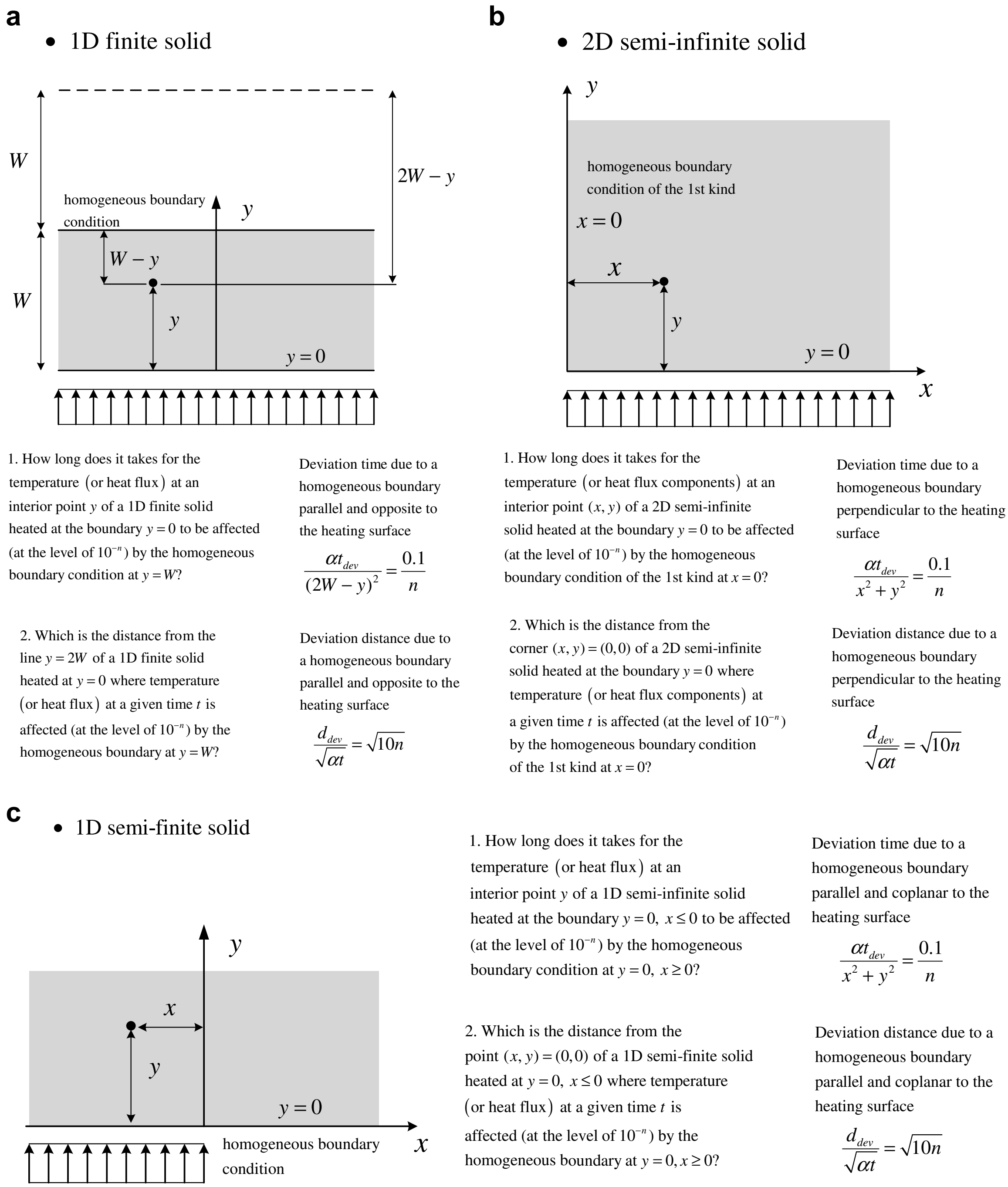

Fig. 2. Deviation times and distances due to a homogeneous boundary condition. a) parallel and opposite ($y = W$) to the heating surface; b) perpendicular ($x = 0$) to the heating surface; and c) parallel and coplanar ($y = 0$, $x \geqslant 0$) to the heating boundary.

If we take the reciprocal of the square root of the deviation time (19) we get a conservative estimate for the deviation distance, that is, the distance $(d_{\text{dev}})_{y=2W}$ measured from the 'virtual' location $y = 2W$ of the thermal disturbance at which its deviation effects just begin to be significant (10^{-n}) for temperature and heat flux at a given time t. Therefore, we can write

$$\frac{(d_{\text{dev}})_{y=2W}}{\sqrt{\alpha t}} = \sqrt{10n} \quad t \geqslant 0 \tag{21}$$

Also, the deviation distance may be defined as the distance $(d_{\text{dev}})_{y=W}$ measured from the 'actual' location $y = W$ of the homogeneous boundary condition of a 1D finite solid heated at $y = 0$ at which temperature and heat flux are just affected at a given time t by the above thermal disturbance (Fig. 2a). Thus, we have

$$\frac{d_{\text{dev}}}{\sqrt{\alpha t}} = \sqrt{10n} - \frac{W}{\sqrt{\alpha t}} \quad t \in [t_{\text{pen}}(W), 4t_{\text{pen}}(W)] \tag{22}$$

5. Deviation time due to a homogeneous boundary perpendicular to the heating surface

To obtain the deviation time, consider a homogeneous rectangular corner $x \geqslant 0$, $y \geqslant 0$, with temperature-independent properties and initially at zero temperature, subject to a step change in either temperature, or heat flux or ambient temperature at $y = 0$ (heating or cooling). The $x = 0$ boundary surface is kept at zero temperature (Fig. 2b). This transient heat conduction problem may concisely be denoted by $X10B0YJ0B1T0$ (J = 1, 2 or 3), where X denotes the x-direction.

The presence of the $x = 0$ homogeneous boundary causes thermal deviation effects inside the rectangular corner. Thus, in the strict sense, the simple 1D semi-infinite solutions cannot be 'used' for calculating temperature and heat flux in the y-direction in the 2D semi-infinite solid. However, if the time is much smaller than the time constant calculated at the point (x,y) of interest, that is, $t \ll (x^2 + y^2)/\alpha$, the deviation effects are expected to be negligible and the above 1D semi-infinite solutions can be 'used' with excellent accuracy.

If the $x = 0$ boundary is thermally insulated, then the $X20B0\quad YJ0B1T0$ (J = 1, 2 or 3) cases are 1D cases and no deviation effects are induced in the corner. As the $X10B0$ ($h_{x=0} \to \infty$)$YJ0B1T0$ and $X20B0$ ($h_{x=0} \to 0$) $YJ0B1T0$ cases bracket the $X30B0YJ0B1T0$ one (constant ambient temperature kept at zero at $x = 0$ with finite values of $h_{x=0}$), the deviation effect caused by the boundary condition of Robin type is less or equal (for only $h_{x=0} \to \infty$) to that caused by the boundary condition of Dirichlet type (i.e., the time to cause a given deviation effect will tend to be longer). Therefore, the $X10B0\ YJ0B1T0$ (J = 1, 2 or 3) cases are the most conservative ones.

To establish a single criterion under which the temperature and heat flux in the y-direction may be calculated inside the rectangular corner through the 1D semi-infinite solutions of Section 2, the deviation effects due to the homogeneous boundary condition of Dirichlet type at $x = 0$ have to be analyzed at any time and point near the corner on both temperature and heat flux components (in the x- and y-directions). These may be analyzed in a complete manner using three dimensionless groupings defined as

$$\begin{aligned} \varepsilon_T(x,y,t) &= \left| \frac{T_{X10B0YJ0B1T0}(x,y,t) - T_{YJ0B1T0}(y,t)}{T_{YJ0B1T0}(0,t)} \right| \\ \varepsilon_{q_x}(x,y,t) &= \left| \frac{q_{x,X10B0YJ0B1T0}(x,y,t)}{q_{y,YJ0B1T0}(0,t)} \right| \\ \varepsilon_{q_y}(x,y,t) &= \left| \frac{q_{y,X10B0YJ0B1T0}(x,y,t) - q_{y,YJ0B1T0}(y,t)}{q_{y,YJ0B1T0}(0,t)} \right| \end{aligned} \tag{23}$$

where $YJ0B1T0$ (J = 1, 2 or 3) denotes the 1D transient heat conduction problems treated in Section 2. The analysis of the deviation effects ε_T, ε_{q_x} and ε_{q_y} defined by Eq. (23) as well as the deviation times deriving from them requires the knowledge of three exact 2D semi-infinite solutions. As they are not available in heat conduction literature, their calculation is explicitly performed in next Sections with the exception of the boundary condition of the 3rd kind which may be taken as

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} + h_{y=0}T(x, y = 0, t) = f(x, y = 0) \tag{24}$$

Concerning this, in fact, it is relevant to note that the $X10B0\quad Y10B1T0$ case ($h_{y=0} \to \infty$ in Eq. (24) with $f(x,y = 0)$ equal to $h_{y=0}T_\infty$ with T_∞ being the ambient temperature) and the $X10B0Y20B1T0$ case ($h_{y=0} \to 0$ in Eq. (24)) bracket the $X10B0Y30B1T0$ one (constant ambient temperature kept at T_∞ with finite values of $h_{y=0}$ at $y = 0$).

6. X10B0Y10B1T0 problem

The exact temperature for the 2D semi-infinite problem here under consideration may be evaluated using Green's functions, that is,

$$T(x,y,t) = T_0\alpha \int_{u=0}^{t} \int_{x'=0}^{\infty} G_{X10}(x,x',u)\mathrm{d}x' \left[-\frac{\partial G_{Y10}}{\partial n'}(y,0,u)\right]\mathrm{d}u \tag{25}$$

where the G_{X10} and G_{Y10} functions are defined through Eqs. (11) for $I = 1$. Thus, we have

$$\begin{aligned} \int_{x'=0}^{\infty} G_{X10}(x,x',u)\mathrm{d}x' &= 1 - \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha u}}\right) \\ -\frac{\partial G_{Y10}}{\partial n'}(y,0,u) &= \frac{y}{\sqrt{4\pi[\alpha u]^3}} \mathrm{e}^{-\frac{y^2}{4\alpha u}} \end{aligned} \tag{26}$$

Substituting Eq. (26) in Eq. (25), we have the temperature field in the corner

$$T(x,y,t) = T_0\left[\mathrm{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right) - \frac{y}{\sqrt{4\pi\alpha}} \int_{u=0}^{t} \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha u}}\right) \frac{1}{u^{3/2}} \mathrm{e}^{-\frac{y^2}{4\alpha u}}\mathrm{d}u\right] \tag{27.1}$$

The evaluation of the above integral gives the so-called I_5– function multiplied by a factor of 2 (Amos, [27]). This function is also treated in App. A of Ref. [22]. Now, differentiating both sides of Eq. (27.1) with respect to x and applying Leibniz's rule, we can get the heat flux $q_x = -k(\partial T/\partial x)$, that is,

$$q_x(x,y,t) = -\frac{2kT_0}{\pi\sqrt{\alpha t}}\left(\frac{y}{\sqrt{\alpha t}}\right)\left(\frac{x^2+y^2}{\alpha t}\right)^{-1} \mathrm{e}^{-\frac{x^2+y^2}{4\alpha t}} \tag{27.2}$$

Also, differentiating both sides of Eq. (27.1) with respect to y and applying the derivative rule of the I_5-function given in Appendix A of Ref. [22], the heat flux $q_y = -k(\partial T/\partial y)$ is

$$q_y(x,y,t) = \frac{kT_0}{\sqrt{\pi\alpha t}} \mathrm{e}^{-\frac{y^2}{4\alpha t}} \left[\mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) + \frac{2}{\sqrt{\pi}}\left(\frac{x}{\sqrt{\alpha t}}\right)\left(\frac{x^2+y^2}{\alpha t}\right)^{-1} \mathrm{e}^{-\frac{x^2}{4\alpha t}}\right] \tag{27.3}$$

6.1. Deviation effects on temperature and heat flux components

Once the exact 2D semi-infinite solutions are known, the deviation effects in the corner may be analyzed. Then, substituting Eqs. (27) in Eqs. (23) as well as the well-known 1D semi-infinite solutions of the $Y10B1T0$ problem [10, p. 150], we obtain

$$\begin{aligned} \varepsilon_T(P,t_{xy}^+) &= \frac{1}{\sqrt{\pi}} \frac{\sqrt{t}}{\sqrt{t_{xy}^+}} \frac{1}{(1+P^2)^{1/2}} I_5\left(\frac{y}{\sqrt{4\alpha}}, \frac{x}{\sqrt{4\alpha}}, \frac{1}{\sqrt{t}}\right) \\ \varepsilon_{q_x}(P,t_{xy}^+) &= \frac{2}{\sqrt{\pi}} \frac{\sqrt{t_{xy}^+}}{(1+P^2)^{1/2}} \mathrm{e}^{-\frac{1}{4t_{xy}^+}} \\ \varepsilon_{q_y}(P,t_{xy}^+) &= \left| \frac{2}{\sqrt{\pi}} \frac{P\sqrt{t_{xy}^+}}{(1+P^2)^{1/2}} \mathrm{e}^{-\frac{1}{4t_{xy}^+}} - \mathrm{e}^{-\frac{1}{4t_{xy}^+}\frac{1}{1+P^2}} \mathrm{erfc}\left[\frac{1}{\sqrt{4t_{xy}^+}} \frac{P}{(1+P^2)^{1/2}}\right] \right| \end{aligned} \tag{28}$$

where

$$\begin{aligned} \frac{x}{\sqrt{4\alpha}} &= \sqrt{t}\left[\frac{1}{\sqrt{4t_{xy}^+}} \frac{P}{(1+P^2)^{1/2}}\right] \\ \frac{y}{\sqrt{4\alpha}} &= \sqrt{t}\left[\frac{1}{\sqrt{4t_{xy}^+}} \frac{1}{(1+P^2)^{1/2}}\right] \end{aligned} \tag{29}$$

Notwithstanding the time coordinate t appears explicitly on the right-hand side of Eq. (28.1), writing out this equation shows that the 2D deviation effects on temperature due to a boundary condition of Dirichlet type depend only on two dimensionless variables, namely $t_{xy}^+ = \alpha t/(x^2 + y^2)$ and $P = x/y$ (or θ as $P = \cot\theta$). As a consequence, when using the Fortran subroutines [27] for computing the I_5-function, it can take $t = 1$ in the computation. Also, the ε_{q_y} case is a bit "tricky" in that for certain *P*'s the term inside the absolute value sign has negative as well as positive values. Values given by Eq. (28) (numerically solved for different values of $\varepsilon = 10^{-n}$ for the dimensionless time $t^+_{xy,dev}$) are shown in both tabular and graphical form for different values of *P*. However, as there exist no deviation effects on temperature and heat flux in the *x*-direction at the "active" boundary surface $y = 0$ ($P \to \infty$), Eqs. (28.1) and (28.2) will not be analyzed at this boundary.

In Table 2 (left side) the values concerning temperature and heat flux in the *y*-direction at $P = 0$ (marked in bold type) are exactly the same as the ones given in Table 1 for $B_y \to \infty$ (Y10) regarding the penetration times. This indicates that the $P = 0$ case of the 2D deviation problem simulates exactly the 1D penetration one with constant surface temperature treated in Section 2. Notice also that the $P = 0$ values for ε_{q_y} may be obtained by analytically solving Eq. (28.3) for the dimensionless time associated with 10^{-n}. In fact, for $P = 0$ this equation reduces to the second of the two Eq. (6). It is also amazing from Table 2 that we have a very weak dependence on *P* for the smallest values of ε. For instance, the $\varepsilon_{q_y} = 10^{-10}$ results from $P = 0$ to infinity only vary from 0.0109 to 0.0144, which is only ±16% different from the average value of 0.0126.

In Fig. 3, however, it was convenient to select $t^+_{xy,dev}$ and θ as non-dimensional variables. This figure refers only to $\varepsilon_T = 10^{-10}$ and indicates that, for $\theta \geqslant 20°$ (i.e. $x \geqslant 2.75y$), the deviation time is less than 0.014.

7. X10B0 Y20B1T0 problem

The exact solutions for the current 2D semi-infinite problem may be derived using Green's functions approach [10]. Thus, we have

$$
\begin{aligned}
T(x,y,t) &= 2q_0 \frac{\alpha}{k} \frac{1}{\sqrt{\pi\alpha t}} \bar{I}_1\left(\frac{y}{\sqrt{4\alpha}}, \frac{x}{\sqrt{4\alpha}}, \frac{1}{\sqrt{t}}\right) \\
q_x(x,y,t) &= -\frac{q_0}{\pi} E_1\left(\frac{x^2+y^2}{4\alpha t}\right) \qquad (30) \\
q_y(x,y,t) &= q_0\left[\operatorname{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right) - \frac{y}{\sqrt{\pi\alpha}} I_5\left(\frac{y}{\sqrt{4\alpha}}, \frac{x}{\sqrt{4\alpha}}, \frac{1}{\sqrt{t}}\right)\right]
\end{aligned}
$$

where the computation of the so-called $\bar{I}_1$–function may be performed using Amos's subroutines [27]; and $E_1(z)$ is the exponential integral [28]. Note that comparing Eqs. (27.1) and (30.3) indicates the heat flux q_y satisfies formally the same differential equation and boundary conditions as $T(x,y,t)$. In fact, $q_y(x,y=0,t) = q_0$ and $q_y(x=0,y,t) = 0$.

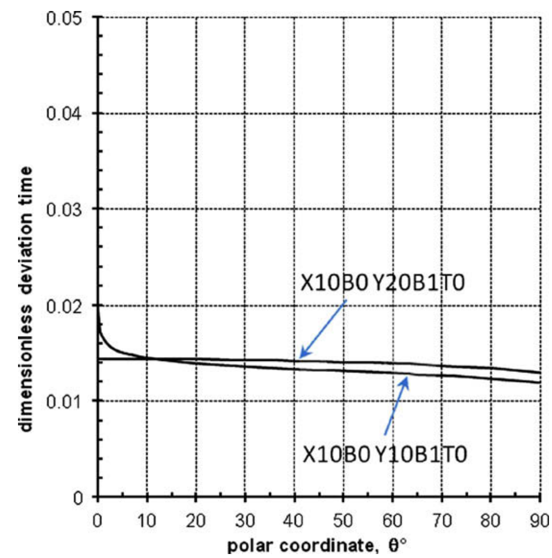

Fig. 3. Dimensionless deviation time $t^+_{xy,\mathrm{dev}}$ as a function of θ ($P \to \infty \Rightarrow \theta = 0°$; $P = 0 \Rightarrow \theta = 90°$) for $\varepsilon_T = 10^{-10}$.

7.1. Deviation effects on temperature and heat flux components

The thermal deviation effects due to the homogeneous boundary condition of the 1st kind at $x = 0$ may be estimated using Eqs. (23) and (30) as well as the well-known 1D semi-infinite solutions of the *Y20B1T0* problem [10, p. 151]. They are shown in both tabular and graphical form for different values of *P*.

In Table 2 (right side) the values regarding temperature and heat flux in the *y*-direction at $P = 0$ (marked in bold type) are exactly the same as the ones given in Table 1 for $B_y = 0$ (Y20). This result indicates that the $P = 0$ case of the 2D deviation problem simulates exactly the 1D penetration one with constant surface heat flux treated in Section 2. Also, it is absolutely amazing from Table 2 that we have a very weak dependence on *P* for the smallest values of ε . For instance, the $\varepsilon_T = 10^{-10}$ findings from $P = 0$ to infinity only vary from 0.0130 to 0.0144, which is only ±5% different from the average value of 0.0137. For the case $\varepsilon_{q_x} = 10^{-10}$, the dimensionless time of 0.0132 is even independent of *P*.

Fig. 3 shows the dimensionless time t^+_{xy} versus θ for only $\varepsilon_T = 10^{-10}$. It indicates that, for all the θ values, the deviation time is always less than 0.015.

Table 2
Dimensionless deviation times $t^+_{xy,\mathrm{dev}}$ for the thermal deviation effects ε_T, ε_{q_x} and ε_{q_y} to reach different small numbers 10^{-n} for different values of *P*

P	*X10B0Y10B1T0*			*X10B0Y20B1T0*		
	10^{-2}	10^{-4}	10^{-10}	10^{-2}	10^{-4}	10^{-10}
Thermal deviation effects on temperature (ε_T)						
0	**0.0754**	**0.0330**	**0.0120**	**0.0970**	**0.0387**	**0.0130**
0.5	0.1029	0.0382	0.0129	0.1324	0.0443	0.0139
1	0.1323	0.0415	0.0133	0.1702	0.0466	0.0142
∞	–	–	–	–	0.0488	0.0144
Thermal deviation effects on heat flux in the x-direction (ε_{q_x})						
0	0.0703	0.0328	0.0119	0.0897	0.0409	0.0132
0.5	0.0728	0.0333	0.0120	0.0897	0.0409	0.0132
1	0.0787	0.0343	0.0121	0.0897	0.0409	0.0132
∞	–	–	–	0.0897	0.0409	0.0132
Thermal deviation effects on heat flux in the y-direction (ε_{q_y})						
0	**0.0543**	**0.0271**	**0.0109**	**0.0754**	**0.0330**	**0.0120**
0.5	0.0776	0.0315	0.0117	0.1029	0.0382	0.0128
1	0.2741	0.0353	0.0122	0.1323	0.0415	0.0133
∞	0.1348	0.0477	0.0144	–	–	–

8. Summary of the XI0B0 YJ0B1T0 problems

From what has been said in the previous Sections, the problem denoted by *XI0B0Y10B1T0* with the heat flux in the *y*-direction at $P = 0$ is the most conservative one among all the *XI0B0YJ0B1T0* problems (*I, J* = 1, 2, 3). In particular, the $P = 0$ case of the 2D deviation problem for both temperature and heat flux in the *y*-direction simulates exactly the 1D penetration problem with a step change in temperature or heat flux (or a heat pulse at time zero) at the boundary. Also, it simulates exactly the 1D penetration problem related to a plane heat source with a heat pulse at time zero or a source term variable in time as $1/\sqrt{t}$. In addition, it is relevant to note that the deviation time is in general only a weak function of *P* for both the temperature and heat flux components except the region very close to the heated surface ($P \gg 1$). For the case denoted *XI0B0 Y10B1T0* the deviation time based on the heat flux in the *x*-direction is even independent of *P*.

Therefore, the dimensionless deviation time of $0.1/n$ (where 0.1 is a conservative value for $[4\ln(10)]^{-1} \cong 0.109$) is the smallest value for any *P* of the corner at the indicated level of 10^{-n}. Then, the deviation time may be defined as the time that it takes for the temperature (or heat flux components) at a point (x,y) in the *XI0B0YJ0B1T0* (I, J = 1, 2 or 3) problem to deviate from the *YJ0B1T0* (J = 1, 2 or 3) problem about one part in 10^n compared to the temperature (or heat flux in the *y*-direction) at $y = 0$ at the same time, that is,

$$\frac{\alpha t_{\mathrm{dev}}}{x^2+y^2} \cong \frac{0.1}{n} \quad (x \geqslant 0, y \geqslant 0, t > 0) \tag{31}$$

where (x^2+y^2) is the square of the distance from the point $(x,y) = (0,0)$ to the point of interest (x,y). When $n = 2$, we have the 'visual' deviation time. (For $I = 2$, the boundary surface at $x = 0$ is thermally insulated and, hence, there is no deviation). Contrary to what happens in the penetration times (8) and (17) and according to what happens in the deviation time (19), the length $\sqrt{x^2+y^2}$ appearing in the deviation time (31) is greater than the distance between the 'inactive' thermal disturbance (homogeneous boundary condition at $x = 0$) and the point of interest, namely *x*, as shown in Fig. 2b. Therefore, this length may be considered as the distance between the 'virtual' location $(x,y) = (0,0)$ of the thermal 'inactive' disturbance (homogeneous boundary at $x = 0$) and the actual location (x,y) of the point of interest.

As the most rapid variation would be for an uniform sudden change in temperature at the heated surface $y = 0$ (most conservative case), Eq. (31) gives the smallest penetration time. Therefore, it covers any time and space variation of temperature, heat flux or ambient temperature at this boundary (for a given maximum surface temperature, heat flux or ambient temperature), including the heat pulse at $t = 0$.

Now, the deviation time (31) may be split into two components: (1) the penetration time calculated at *y* and (2) the co-penetration time related to the homogeneous boundary condition at $x = 0$ and perpendicular to the heating surface. Therefore, we can write

$$t_{\mathrm{dev}} \cong \underbrace{\frac{0.1}{n\alpha}y^2}_{t_{\mathrm{pen}}(y)} + \underbrace{\frac{0.1}{n\alpha}x^2}_{t_{co-pen}(x)} \tag{32}$$

The length which appears in the co-penetration time is the distance *x* between the thermal disturbance located at $x = 0$ and the point of interest. The so-called "finite-space time" appearing in Eq. (20) is here absent due to the infinite dimension of the 2D solid.

If we take the reciprocal of the square root of the deviation time (31) we get a conservative estimate for the deviation distance, that is, the distance $(d_{\mathrm{dev}})_{(x,y)=(0,0)}$ measured from the 'virtual' location $(x,y) = (0,0)$ (corner) of the thermal disturbance at which its deviation effects just begin to be significant (10^{-n}) for temperature and heat flux components at a given time *t*. Therefore, we can write

$$\frac{(d_{\mathrm{dev}})_{(x,y)=(0,0)}}{\sqrt{\alpha t}} = \sqrt{10n} \quad t \geqslant 0 \tag{33}$$

Also, the deviation distance may be defined as the distance $(d_{\mathrm{dev}})_{x=0}$ measured from the 'actual' location $x = 0$ of the homogeneous boundary condition of a 2D semi-infinite solid heated at $y = 0$ at which temperature and heat flux components are just affected at a given time *t* by the above thermal disturbance (Fig. 2b). Thus, we have

$$\frac{(d_{\mathrm{dev}})_{x=0}}{\sqrt{\alpha t}} = \sqrt{10n - \frac{y^2}{\alpha t}} \quad t \geqslant t_{\mathrm{pen}}(y) \tag{34}$$

9. Deviation time due to a homogeneous boundary parallel and coplanar to the heating surface

To obtain the deviation time in this case, consider a semi-infinite Cartesian body $y \geqslant 0$ with temperature-independent properties. The initial temperature is zero. At time $t = 0$, heating begins over half ($x \in (-\infty, 0]$) of the boundary surface $y = 0$, as shown in Fig. 2c, and it can be of any kind (1st, 2nd or 3rd). This is the *X00YI0Bx5T0* (I = 1, 2 or 3) case, where "*x*5" denotes a step change in the *x*-direction.

To establish a single criterion under which temperature and heat flux may be calculated inside the rectangular domain $x \leqslant 0$, $y \geqslant 0$, through the well-known 1D semi-infinite solutions of Section 2, the deviation effects due to the homogeneous boundary at $y = 0$, $x \geqslant 0$ (i.e., parallel and coplanar to the heating boundary at $y = 0$, $x \leqslant 0$) have to be analyzed. This may be done using three dimensionless groups defined as

$$\begin{aligned} \varepsilon_T(x,y,t) &= \left|\frac{T_{X00YI0Bx5T0}(x,y,t) - T_{YI0B1T0}(y,t)}{T_{YI0B1T0}(0,t)}\right| \\ \varepsilon_{q_x}(x,y,t) &= \left|\frac{q_{x,X00YI0B1T0}(x,y,t)}{q_{y,YI0B1T0}(0,t)}\right| \quad (x \leqslant 0, y \geqslant 0) \\ \varepsilon_{q_y}(x,y,t) &= \left|\frac{q_{y,X00YI0B1T0}(x,y,t) - q_{y,YI0B1T0}(y,t)}{q_{y,YI0B1T0}(0,t)}\right| \end{aligned} \tag{35}$$

where *YI0B1T0* (I = 1, 2 or 3) denotes the 1D transient heat conduction problems treated in Section 2. The analysis of the deviation effects ε_T, ε_{q_x} and ε_{q_y} defined by Eqs. (35) as well as the deviation times deriving from them requires the knowledge of three exact 2D semi-infinite solutions. Because of space limitations, only the *I = 2* case solved in Ref. [10, p. 183] will be considered here. The temperature solution is

$$T(x,y,t) = \frac{q_0}{k}\left(\frac{\alpha}{\pi}\right)^{1/2}\frac{1}{2}\int_{u=0}^{t}\frac{1}{\sqrt{u}}\mathrm{e}^{-\frac{y^2}{4\alpha u}}\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha u}}\right)\mathrm{d}u \tag{36}$$

where the time integral can be evaluated in closed form using an Amos's function denoted by $2I_1^c\left(\frac{y}{\sqrt{4\alpha}}, \frac{x}{\sqrt{4\alpha}}, \frac{1}{\sqrt{u}}\right)$. (See page 1.0-3 and 2.6.1 of Ref. [27]). Eq. (36) simplifies for the special cases of $x = 0$ and $y = 0$, as shown in [10].

9.1. Deviation effects

The thermal deviation effects on temperature due to the homogeneous boundary condition of the 2nd kind at $y = 0$, $x \geqslant 0$, may be estimated using Eqs. (35.1) and (36) as well as the well-known 1D semi-infinite solution of the *Y20B1T0* problem.

In the current case, it may be proven that the dimensionless deviation time of $0.1/n$ is again the smallest value for any $P = y/x$ of the rectangular domain $x \leqslant 0$, $y \geqslant 0$, at the indicated level of 10^{-n}. Then, the deviation time may be defined as the time that it takes for the temperature and heat flux to be just disturbed (one part in 10^n) by the homogeneous boundary condition parallel and coplanar to the heating surface. Eqs. (31) and (33) applies to this case too. In particular, the distance $\sqrt{x^2+y^2}$ may be considered as the distance between the 'virtual' location $(x,y) = (0,0)$ of the thermal 'inactive' disturbance (homogeneous boundary at $y = 0$, $x \geqslant 0$) and the actual location (x,y) of the point.

10. Criterion for the diffusion of thermal disturbances

In the previous sections two characteristic times have been defined with reference to 1D and 2D Cartesian solids heated (or cooled) through a boundary surface or a local heat source. The similarity of the results obtained from them for different cases allows us to define a single criterion able to model the diffusion of the thermal disturbances inside a 2D Cartesian heat-conducting body. This criterion may be taken as

$$\frac{\alpha t_{\text{dist}}}{d^2} = \frac{0.1}{n} \quad (37)$$

It gives the time that it takes for a generic thermal disturbance at a point of a solid to reach another point of the same solid at the level of one part in 10^n. The d denotes the distance between the two points. In detail, the location of the thermal disturbance is the actual location for an 'active' disturbance but it is the 'virtual' location for an 'inactive' one (Sections 4, 8 and 9). Also, the reciprocal of the square root of Eq. (37) gives the distance $(d_{\text{dist}}/\sqrt{\alpha t}) = \sqrt{10n}$ between the point ('actual' or 'virtual') of disturbance ('active' or 'inactive', respectively) and the point of interest where the level of the disturbance at a given time t is reduced to one part in 10^n ($n = 1,2,\ldots$). For $n = 2$, we have the 'visual' disturbance distance.

Although Eq. (37) has been obtained for the simple cases denoted by *YI0B1T0*, *XI0B0YJ0B1T0* and *X00YI0Bx5T0* (*I*, *J* = 1, 2 or 3), it may be extended to whatever type of boundary condition (also arbitrary in space and time), volume heat source and initial temperature distribution we have. In fact, when heating a solid through a boundary or heat source, the most rapid variation would be for the uniform and local sudden change in conditions analyzed in the paper. Similarly, the smallest disturbance time would be for the uniform initial thermal field considered in the present treatment.

Eq. (37) is not only insensitive to various boundary conditions and 2D conditions, but it is also relatively insensitive to the % deviation. In fact, decreasing the deviation from 10^{-2} to 10^{-10}, a factor of 10^8 results in a decrease in the criterion only be a factor of 5.

11. Numerical example

Consider a rectangle $0 \leqslant x \leqslant L = 2W$, $0 \leqslant y \leqslant W$, initially at zero temperature and thermally insulated at the boundaries $x = 0$ and $x = L$, as shown in Fig. 4. Also, we have a zero temperature at $y = W$ and a step change in the heat flux in the x-direction at the $y = 0$ surface. In particular, the heat flux is constant with time and other than zero only over the region $0 \leqslant x \leqslant L_1 = 3W/2$. This 2D transient problem may be denoted by *X22B00Y21B*(*x*5)*0T0*, where "(*x*5)" denotes a step change in the x-direction.

Suppose now that the only thermal region of interest is near the corner $(x,y) = (0,0)$ of the rectangle and that the dimensionless time of interest falls in the range $0 \to t_W^+$, where $t_W^+ = \alpha t/W^2$. In virtue of the criterion (37), we need not consider the complete rectangle but only a part of it. The extension of this part may be evaluated applying Eq. (37) and it depends on the accuracy desired and times of interest.

The starting point is to characterize the thermal disturbances diffusing in the region of interest. In the current problem, two thermal 'inactive' disturbances are present:

- The former is due to the homogeneous boundary condition of the 1st kind at $y = W$. The distance measured from the 'virtual' location $y = 2W$ at which its deviation effects just begin to be significant (10^{-n}) for temperature and heat flux at a given time t is $(d_{\text{dev}})_{y=2W} = W\sqrt{10nt_W^+}$.
- The latter is caused by the homogeneous boundary condition of the 2nd kind at $y = 0$, $3W/2 \leqslant x \leqslant 2W$. The distance measured from the 'virtual' location $(x,y) = (3W/2,0)$, where the level of the disturbance is reduced to 10^{-n}, is given by $(d_{\text{dev}})_{y=2W} = W\sqrt{10nt_W^+}$.

Now, if $t_W^+ = 0.025$ (small dimensionless time) and the accuracy desired is of 10^{-4} (i.e., $n = 4$), then both the disturbance distances

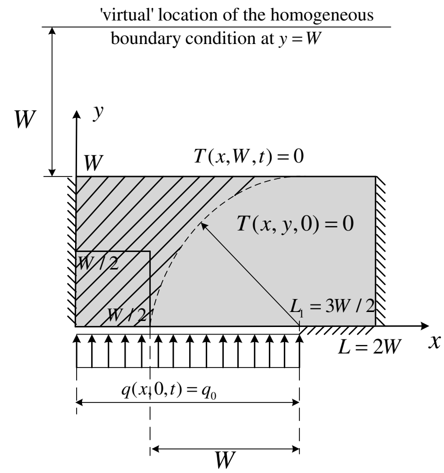

Fig. 4. Transient heat conduction problem in a rectangle initially at zero temperature with insulated boundaries in the x-direction, zero prescribed temperature at $y = W$ and a time-step change in the heat flux at $y = 0$ but only over the region $0 \leqslant x \leqslant L_1 = 3W/2$.

are of W. This indicates that, for $t_W^+ \leqslant 0.025$, any point of the area shown shaded in Fig. 4 is reached by both the thermal disturbances at $y = 0$, $3W/2 \leqslant x \leqslant 2W$ and $y = W$ with deviation effects less than or equal to one part in 10^4. Therefore, to compute the solution in the rectangular area of interest, for example $0 \leqslant x \leqslant W/2$, $0 \leqslant y \leqslant W/2$, and in the time interval of interest, namely $0 \leqslant t^+ \leqslant 0.025$, we can consider only this thermal region in place of considering the whole rectangle. Assuming an insulation condition at $x = W/2$ and $y = W/2$, the initial 2D transient problem reduces to the sub-problem denoted by *X22B00Y22B00T0*. (The insulation condition is one that does not "initiate" a disturbance in temperature.)

From what was said, the solution will be computed approximately but with absolute errors less than 10^{-4}. However, as the thermal area ($W^2/4$) is eight times smaller than the primary area ($2W^2$), a substantial saving of computational resources when using numerical methods is obtainable. Also, there can be advantages for analytical methods. For example, the number of terms in the 2D summations needed to obtain the long-time solution (proportional to L by W [22, p. 4249]) may significantly be reduced. In the current problem, in fact, it reduces significantly (87.5%). In addition, as the time-partitioning method [10] provides an alternative but computationally more efficient and accurate way to get steady state solutions [22] than the well-known SOV method, the reduction of 87.5% can be further effective.

12. Conclusions

The analysis of both the penetration and deviation effects in a 2D Cartesian homogeneous domain has allowed us to define a single criterion modeling the diffusion of 'active' and 'inactive' thermal disturbances at a level of one part in 10^n. It was found that the penetration and deviation times depend linearly on the square of the distance between the point of interest and the point of disturbance through a factor, i.e. $0.1/n$, which is a function of the desired level of disturbance.

As regards the deviation effects due to a homogeneous boundary perpendicular to the heating one, the analysis of their diffusion has required to explicitly solve (using Green's functions) two complex transient 2D semi-infinite problems in that their solutions were not available in literature. It was found that the deviation time is in general only a weak function of the space coordinate ratio having 0.01 as an extremely conservative value. This value is associated with a disturbance of 1 part in 10^{10} of the surface conditions. We have noted that having a criterion of 1 part in 10^4 results in an increase from 0.01 to about 0.025; that shows the insensitivity of the criterion since 10^4 and 10^{10} (though different by a factor of 10^6) change the criterion less that a factor of 3.

The application of the diffusion criterion is important because it allows the prediction, at a region of interest, of the thermal disturbances caused by some distant parts of a heat conducting homogeneous body. In such a way, we need not consider the complete domain with a substantial saving of computational resources for numerically based methods when a semi-infinite or "large" body is considered. However, there are also advantages for analytical methods, in particular related to the reduction of the number of terms needed to obtain long-time solutions when the separation of variables (SOV) method is applied.

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